Strong Graphs, Its Properties and some Families of Strong Graphs

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Abstract— Graph labeling has wide applications in Radar, Communication networks,Circuit design,Coding Theory,Astronomy,X-ray,Crystallography,Database Management and Modelling of Constraint Programming over Finite domain.In this paper Strong graphs is introduced. Necessary and Sufficient Condition for existence of strong α labeled graphs is proved. The properties of strong α labeled graphs is derived.Introduced a category of strong Graphs called Nishad Graph the complement of which fails to be strong. Also a family of strong graphs is discovered.

Keywords— Graph labeling, Nishad Graph, Strong edge, Strong Graphs.

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1 Introduction

ROSA defined α labeling of a graph in 1966. In 1992 Gallian weakened the condition and introduced weakly α labeling.Analyzing importance of edges having some specified property in application point of view, In this paper I strengthened the condition and introduced strong α labeled graphs. Generalazing the concept to all types of graph labeling, a family of strong graphs is discovered.

2 Section I: Preliminaries

Definition 2.1: Graph labeling
A labeling of a Graph G=(V,E) is a one to one mapping ψ of the vertex set V(G) in to the non negative set of integers that induces for each edge e =uv ϵE(G), a label depending on the vertex labels ψ(u) and ψ(v).

Definition 2.2: Strong Graph
A labeled graph G=(V,E) is a strong graph if it satisfies the condition that there exist a number δ where 0 < δ < Max {ψ(e)/e ϵ E(G) and ψ, the labeling} such that Min{ψ(u), ψ(v)}< ψ(uv) < Max {ψ(u), ψ(v)}
Note: 2.2.1: If the labeling is α then Max {ψ(e)/e ϵ E(G)}= ∣ E(G)∣ and ψ(uv) = ∣ ψ(v) – ψ(u) ∣

Definition 2.3: Strong edge
A strong edge e =uv of a strong graph is the edge which satisfies Min{ψ(u), ψ(v)}< ψ(uv) < Max {ψ(u), ψ(v)}.

Definition 2.4: N Graph
An N Graph is a strong α labeled graph N of $|E(N)|=n$ with the following properties.

1. Its $n-1$ edges have a common vertex u where $\psi(u)=n$
2. The nth edge vv'has α labeling $\psi(v)=n-2k$ and $\psi(v^1)=n-\kappa, \kappa\in N$.

For the convenience of proving theories, we shall call N graph by the name of the author.i,e, Nishad Graph

3 Section II: Properties of strong graphs

Theorem 3.1: An α - labeling is strong iff there exists an edge $e=uv\in \mathrm{E}(G)$ such that $0<\psi(u)\leq n/2-1$ and $\psi(v)\geq n/2+\psi(u)$ if n is even, $0<\psi(u)\leq n+1/2-1$ and $\psi(v)\geq n+1/2+\psi(u)$ if n is odd.

Proof :

Part 1 : Assume that α - labeling is strong.Then there exists δ such-that $0<\delta<n$ and

$$Min\{\psi(u),\psi(v)\}<\delta<Max\{\psi(u),\psi(v)\}$$

To prove that there exists an edge $e=uv\in \mathrm{E}(G)$ with the given condition holds

Let $|E(G)|=n$ is even. Let $e=uv\in E(G)$

To prove that $0<\psi(u)\leq n/2-1$ and $\psi(v)\geq n/2+\psi/u)$

Suppose $\psi(u)=0, \forall u\in e=uv\in E(u)$

Since $|E(G)|=n$, there exists $v\in V(G)$ such that

$e=uv$ and $\psi(u)=0$, $\psi(v)=n$

$$\therefore |\psi(u)-\psi(v)|=n$$

i.e, $\delta = n$, which is a contradiction

$\therefore \psi(u) > 0$ for some $u \in v(G) \rightarrow (1)$

Suppose $\psi(u) > n/2 - 1$ and $\psi(v) < n/2 + \psi(u), \forall e = uv$

Now $\psi(v) - \psi(u) < n/2 + \psi(u) - n/2 + 1$

$< \psi(u) + 1$

Since both LHS and RHS are +ve numbers

$$\delta = |\psi(v) - \psi(u)| < \psi(u) + 1 \rightarrow (2)$$

By the assumption, it is clear that

$$Min[\psi(u), \psi(v)] < \delta \rightarrow (3)$$

Case1 Let $Min[\psi(u), \psi(v)] = \psi(u)$

From equations (1), (2) and (3), we get

$$\psi(u) < \delta < \psi(u) + 1$$

$\Rightarrow \delta$ is not an integer .

This is a contradiction.

Case 2 Let $Min[\psi(u), \psi(v)] = \psi(v) \rightarrow (4)$ By assumption $\psi(v) < \delta$, Since $\psi(v) < \psi(u)$, and both are integers $\psi(u) \geq \psi(v) + 1$.

Now $\psi(v) - \psi(u) < n/2 + \psi(u) - n/2 = \psi(u)$

i.e, $\delta = |\psi(v) - \psi(u)| < \psi(u) \ and \ \psi(u) \geq \psi(v) + 1$

i.e, $\psi(v) < \delta < \psi(v) + 1$. $\Rightarrow \delta$ is not an integer .Which contradicts the assumption.

Therefore

$\psi(u) \leq n/2 - 1$ and $\psi(v) \geq n/2 + \psi(u)$

Part ll

Assume that $n = |E(G)|$ is even and the condition $0 < \psi(u) \leq n/2 - 1, \psi(v) \geq n/2 + \psi(u)$ holds for some $e = uv$ Define $\delta = |\psi(v) - \psi(u)|$

Given $\psi(v) \geq n/2 + \psi(u)$ and $\psi(u) \leq n/2 - 1$

ie $\psi(v) - \psi(u) \geq n/2 + \psi(u) + 1 - n/2$

$\geq \psi(u) + 1$

ie $\psi(v) - \psi(u) \geq \psi(u) + 1 > \psi(u) > 0$

$$\Rightarrow \delta = |\psi(v) - \psi(u)| > 0 \rightarrow (5)$$

Since

$\psi(u), \psi(v) \in \{0, 1, 2...., n\}, Max\{|\psi(v) - \psi(u)|\} = n$

ie $\delta \leq n$

But it is given that $\psi(u) > 0$

ie, $\psi(u) \in \{1, 2, 3, ..., n\}$

$\therefore |\psi(v) - \psi(u)| \leq n - 1$

$$\Rightarrow \delta < n \rightarrow (6)$$

From equations (5) and (6) $o < \delta < n$

To show that there exists a δ

such that $Min[\psi(u), \psi(v)] < \delta < Max[\psi(u), \psi(v)]$

Given $0 < \psi(u) \leq n/2 - 1$ and $\psi(v) \geq n/2 + \psi(u)$

Consider $\psi(u) = n/2 - 1$, where n is even and $n > 2$

and $\psi(v) = n/2 + \psi(u)$

Now $\psi(u) < \psi(v)$

$$Min[\psi(u), \psi(v)] = \psi(u)$$

$$\delta = |\psi(v) - \psi(u)| = n/2 + \psi(u) - n/2 + 1 = \psi(u) + 1$$

$$Max[\psi(u), \psi(v)] = \psi(v) = n/2 + \psi(u), \ n \text{ is even}, n > 2$$

Its clear that $Min[\psi(u), \psi(v)] < \delta < Max[\psi(u), \psi(v)]$

Similarly we shall prove Part l and Part ll for odd n. Hence the proof.

Theorem 3.2:

Set of all strong α labeled graphs are closed under in version

Proof : It is enough to prove that the inverse of a strong α labeling is strong α labeling.

Assume that the α labeling ψ is strong.

$\therefore$ There exists δ such that $0 < \delta < n$ and $Min[\psi(u), \psi(v)] < \delta < Max[\psi(u), \psi(v)] \rightarrow (7)$

To prove that the inverse ψ^{-1} is strong.

ψ^{-1} is defined as $\psi^{-1}(u) \equiv \delta - \psi(u) (\text{mod}\, n+1)$

Since ψ is strong $\psi^{-1}(u) \equiv \delta - \psi(u) (\text{mod}\, n+1), 0 < \delta < n$

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$$Min[\psi^{-1}(u),\psi^{-1}(v)] =$$
$$Min[\delta-\psi(u)(\bmod n+1), \delta-\psi(v) \bmod(n+1)]$$
$$= Min[-\psi(u)(\bmod n+1), -\psi(v)(\bmod n+1)]+\delta$$
$$= Min[-\psi(u), -\psi(v)](\bmod n+1)+\delta$$
$$= -Max[\psi(u),\psi(v)](\bmod n+1)+\delta$$

Similarly we shall show that

$$Max\{\psi^{-1}(u),\psi^{-1}(v)\} = -Min\{\psi(u),\psi(v)\}(\bmod n+1)+\delta$$
$$\therefore Min\{\psi^{-1}(u),\psi^{-1}(v)\}-\delta =$$
$$Max\{\psi(u),\psi(v)\}(\bmod n+1)$$

$$Max\{\psi^{-1}(u),\psi^{-1}(v)\}-\delta = -Min\{\psi(u),\psi(v)\}(\bmod n+1)$$

By equation (7), we have

$$0<\delta<n$$

Such that $Min\{\psi(u),\psi(v)\}<\delta<Max\{\psi(u),\psi(v)\}$

$$\Rightarrow -Min\{\psi(u),\psi(v)\} > -\delta > -Max\{\psi(u),\psi(v)\}$$
$$\Rightarrow -Min\{\psi(u),\psi(v)\}(\bmod n+1) > -\delta(\bmod n+1) > -Max\{\psi(u),\psi(v)\}(\bmod n+1)$$
$$\Rightarrow Max\{\psi^{-1}(u),\psi^{-1}(v)\}-\delta > -\delta(\bmod n+1) > Min\{\psi^{-1}(u),\psi^{-1}(v)\}-\delta$$
$$\Rightarrow Max\{\psi^{-1}(u),\psi^{-1}(v)\} > \delta-\delta(\bmod n+1) > Min\{\psi^{-1}(u),\psi^{-1}(v)\}$$

Let $\delta-\delta(\bmod n+1)=\delta^1$

$0<\delta^1<n$ Since $0<\delta<n$

$\therefore \psi^{-1}$ is strong. Hence the proof

Theorem 3.3 : Set of all strong α labeled graphs except the set of Nishad graphs is closed under complementation.

Proof: Part 1.

To prove that the compliment of strong α labeling is strong for the strong graphs except Nishad graph.

Let the α labeling be strong. Then there exists at least one number δ, $0<\delta<n$ such that $Min[\psi(u),\psi(v)]<\delta<Max[\psi(u),\psi(v)]$

Now consider the complement of this α labeling by definition $\psi^C(u)=n-\psi(u)$

To prove that ψ^C is strong α labeling.

It is enough to prove that there exists a number δ, $(0<\delta<n)$ such-that $Min[\psi^c(u),\psi^c(v)]<\delta<Max[\psi^c(u),\psi^c(v)]$ .Let $\psi(u)=k_1$ and $\psi(v)=k_2$ Also let $k_1<k_2$ then from the given condition we get $k_1<\delta<k_2 \rightarrow (8)$.

Now

$$\psi^c(u)=n-k_1$$
$$\psi^c(v)=n-k_2$$
$$k_1<k_2, n-k_1>n-k_2$$

Therefore $Min\{\psi^{\subset}(u),\psi^{\subset}(v)\}=Min\{n-k_1,n-k_2\}$

$$=n-k_2$$

Similarly $Max\{\psi^{\subset}(u),\psi^{\subset}(v)\}=n-k_1$

Since $n-k_2<n-k_1$, by By property of real numbers there exists $\delta_1\in k$ such that $n-k_2<\delta_1<n-k_1 \rightarrow (9)$,

To prove that $\delta_1\in\{1,2,3,\ldots\ldots,n-1\}$

Define $\delta_1=(n-k_1)-(n-k_2)=k_2-k_1$

$$=\psi(v)-\psi(u)$$

Then equation (9) holds and $\delta_1\in\{0,1,2,\ldots\ldots,n\}$

Since $\psi(u)-\psi(v)\in\{0,1,2,\ldots\ldots,n\}$

Now to prove that $\delta_1\neq 0 \, and \, \delta_1\neq n$

Case1 : Let

$$\delta_1=0$$
$$\Rightarrow k_2-k_1=0$$
$$\Rightarrow k_2=k_1$$

which is a contradiction to equation (8)

Case2 :

Let

$$\delta_1=n$$
$$k_2>k_1 and$$
$$k_2,k_1\in\{0,1,\ldots..n\}$$
$$\Rightarrow k_2=n, k_1=0$$

$\therefore$ By equation (9) $\quad 0<\delta_1<n \Rightarrow 0<n<n$

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Which is a contradiction . Hence the proof for part I

Part II.

The Complement of Nishad Ggraph is not strong. As per definition of Nishad Graph ,it has a strong α labeling and (n-1) edges have common vertex u, such that $\psi(u) = n$ and the n[th] edge vv[1] has α labeling $\psi(v) = n-2,\ \psi(v^1) = n-k$

Consider the complement of Nishad Ggraph

$$\psi^{\subset}(u) = n - n = 0$$
$$\psi^{\subset}(v) = n - (n-2k) = 2k$$
$$\psi^{\subset}(v^1) = n - (n-k) = k$$

Since n-1 edges have common vertex u, for all edges $uu^1 \in E(G), \psi^c(u) = 0$ and $\psi^c(u^1) = r_i$ where

$ri \in \{1,2,3...,n\} - \{k\}, i = 1,2,3...,n-1$

$\therefore$ For all $uu^1 \in E(u), \delta = \left|\psi^{\subset}(u) - \psi^{\subset}(u^1)\right| = ri$

$\therefore$ These edges $uu^1 \in E(u)$, are not strong edges. The graph will be strong only if vv'is strong but δ for vv' is

$$\delta = \left|\psi^{\subset}(v) - \psi^{\subset}(v^1)\right|$$
$$= |2k - k|$$
$$= k$$

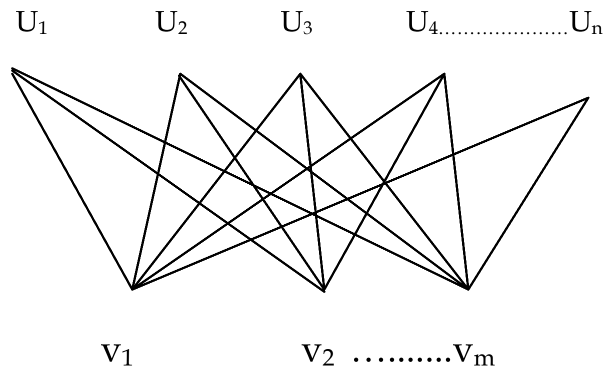
Now $\psi^{\subset}(v^1) = \delta = k \therefore vv^1 is$ not strong ie; The complement of Nishad graph doesnot contain a strong edge. Hence the complement of Nishad graph is not strong.i.e, The set of Nishad Graph is not closed under complementation.

4 SECTION III: FAMILIES OF STRONG GRAPHS

Theorem 4.1: Every even Graceful complete Bipartite graph $K_{m,n}$ for $m,n > 2$ is strong.

Proof : Let $G = K_{m,n}$ with m+n vertices and m,n edges.

Let $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_n$ are the vertices.



Label
$$v_i = 2(i-1), i = 1,2,.......m$$
$$u_j = 2m(n-j+1), j = 1,2,....n$$
$$e_{i,j} = v_i - u_j, \forall_{i,j}$$

This gives even graceful graph labeling to $K_{m,n}$

To prove that $K_{m,n}$ is strong

It is enough to prove that there exists a strong edg.Or it is enough to prove that there exists an edge $e_{i,j}$ such that

$$Min\{\psi(v_i), \psi(u_j)\} < \delta < Max\{\psi(v_i), \psi(u_j)\}$$

Case 1: m and n are even , Consider the edge $e_{m/2, n/2,}$

$$\left|e_{m/2, n/2,}\right| = \left|U_{n/2,} - V_{m/2,}\right|$$
$$= \left|2m(n - n/2, +1) - 2(m/2, -1)\right|$$
$$= |mn + m + 2|$$
$$= mn + m + 2$$

As per even graceful labeling

$$V_{m/2} = m - 2$$
$$U_{m/2} = mn + 2m$$

Now $m - 2 < mn + m + 2 < mn + 2m$ holds when m, n>2

Case 2: m is even and n is odd , Consider the edge $e_{m/2, n+1/2,}$

$$\left|e_{m/2, n+1/2}\right| = \left|U_{n+1/2} - V_{m/2}\right|$$
$$= |m(n+1) - (m-2)|$$
$$= |mn + 2|$$
$$= mn + 2$$

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As per even grceful labeling

$$V_{m/2} = m-2$$
$$U_{n+1/2} = m(n+1)$$

Now $m-2 < mn-2 < m(n+1)$ holds when m, n ≥ 2

Case 3: m is odd and n is even , Consider the edge $e_{m+1/2, n/2,}$

As per even graceful labeling

$$V_{m+1/2} = 2\{\frac{m+1}{2} - 1\} = m-1$$
$$U_{n/2} = m(n+2)$$
$$\left|e_{m+1/2, n/2}\right| = \left|m(n+2) - m + 1\right|$$
$$= \left|mn + m + 1\right|$$
$$= mn + m + 1$$

Now $m-1 < mn+m+1 < m(n+2)$ holds when m ≥ 2, n ≥ 1

Case 4: m is odd and n is odd,

Consider the edge $e_{m+1/2, n+1/2,}$

As per even graceful labeling

$$V_{m+1/2} = m-1$$
$$U_{n+1/2} = m(n+1)$$
$$\left|e_{m+1/2, n+1/2}\right| = \left|m(n+1) - m + 1\right| = mn+1$$

$m-1 < mn+1 < m(n+1)$ holds when $m \geq 2, n \geq 1$

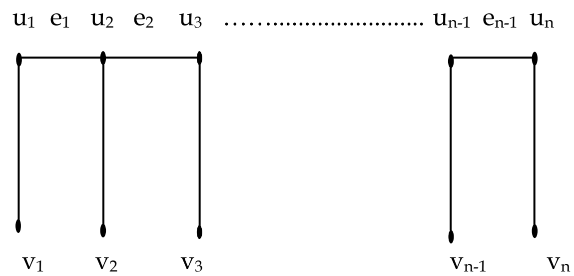
In general even graceful$K_{m,n}$ is strong for $m,n > 2$.

Theorem 4.2: Every even graceful comb graph $G = P_n \Theta K_1$, is strong for even n

Proof: It is enough to prove that there exists a strong edge in G.It is enough to prove that,

There exists $\delta = \left|\psi(u) - \psi(v)\right|$ such that $0 < \delta < n$ and $\min\{\psi(u), \psi(v)\} < \delta < \max\{\psi(u), \psi(v)\}$

Let $G = P_n \Theta K_1$ be a comb with 2n vertices and 2n-1 edges where $n \in E$



Let ei' be the edge connecting the vertices $Ui - Vi$ and let e_i be the edge connecting $U_i - U_{i+1}$ Define

$$\psi(u_{2i-1}) = 4(n-i)+2, \qquad i = 1, 2, \ldots\ldots\ldots\ldots n/2$$
$$\psi(u_{2i}) = 4i-2, \qquad i = 1, 2, \ldots\ldots\ldots\ldots n/2$$
$$\psi(v_{2i-1}) = 4(i-1), \qquad i = 1, 2, \ldots\ldots\ldots\ldots n/2$$
$$\psi(v_{2i}) = 4(n-1), \qquad i = 1, 2, \ldots\ldots\ldots\ldots n/2$$

and label edges

$$e_i = 4(n-i), i = 1, 2, \ldots\ldots\ldots\ldots n$$
$$e_i^1 = 4(n-i)+2, i = 1, 2, \ldots\ldots\ldots n$$

This gives even graceful labeling,

Now consider the edge $e_{n/2,}$

$$\left|e_{n/2}\right| = 4(n - n/2) = 2n$$
$$\psi(U_{2n/2-1}) = 4(n - n/2) + 2 = 2n+2$$
$$\psi(U_{2n/2}) = 4\,n/2 - 2 = 2n-2$$

Now $\delta = \left|\psi(U_{2n/2} - 1) - \psi(U_{2n/2})\right| = \left|2n+2-(2n-2)\right|$

=4

Now $2n-2 < 4 < 2n+2$ holds for $n \geq 2$ Hence the proof.

Theorem 4.3: Odd Graceful Shadow graph $D_2(P_n)$ is strong for $n \geq 5$

Proof: Let G be $D_2(P_n)$with 2n vertices and 4(n-1)edges. Let $v_1, v_2, \ldots\ldots v_n$ be the vertices of first copy of path P_n and $v_{1'}, v_{2'}, \ldots\ldots v_{n'}$ be the vertices of the second copy of path P_n.

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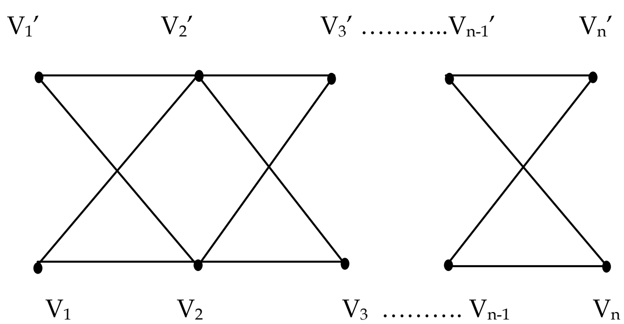
Define

$$f: V(G) \to \{0,1,2,..........2q-1\},\ such\ that$$
$$f(Vi) = 4(i-1),\ i\ is\ odd$$
$$= 4(2n-i)-1,\ i\ is\ even$$
$$f(v_i^1) = 4(i-1)+2,\ i\ is\ odd$$
$$= 4(2n-2)-5,\ i\ even$$

This gives odd graceful labeling for $D_2(P_n)$

To Prove that odd gracefulD$_2$(P$_n$) is strong for $n \geq 5$



Consider the edge V_iV_{i+1},

$$f(V_i) = 4(i-1),\ i\ is\ odd$$
$$f(V_{i+1}) = 4(2n-i),\ i+1\ is\ even$$
$$\delta = \left| f(V_{i+1}) - f(V_i) \right| = \left| \delta n - 4i - 5 - 4i + 4 \right|$$
$$= \delta(n-i)-1$$

Now $4(i-1) < \delta(n-i)-1 < 4(2n-i)-5$ holdsfor $n \geq 5$.

Hence the proof.

5 CONCLUSIONS

The strong graph is discovered .Proved that the set of strong α labeled graphs is closed under inversion but not in complementation. The family of strong graphs is discovered.The N graph having some properties is discovered and provedthat its complement is not strong.The application of N graph is under investigation.

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